Representation Theory Exercise Sheet 1

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MATH0073

These questions cover roughly the first two weeks of lectures and are grouped by the relevant sections. The four questions with a boxed numbers, e.g. 8., are to be handed in during the lecture on Wednesday week 3 (29/01/2020). This excludes any starred parts, which are non-assessed.

1 Sections 1+2

1. Calculate the minimal and characteristic polynomials of

$$\begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

   Find their eigenvalues and eigenvectors. Write the matrices in Jordan canonical form.

2. Calculate matrices defining the representation of D_8, the dihedral group of order 8, given by symmetries of a square.

3. Show, for any representation $\rho\colon G \to \mathrm{GL}(V)$, that the fixed points

$$V^G := \{v \in V \mid \rho(g)v = v \quad \forall g \in G\}$$

   is a sub-vector space of V. Moreover, show that V^G is a subrepresentation isomorphic to some number of copies of $\mathbb{1}$.

4. Show that for any ρ_1, ρ_2, then $\rho_1 \oplus \rho_2$ is isomorphic to $\rho_2 \oplus \rho_1$.

5. Let k be a field of characteristic $\neq 2$ and let $V_2 \subset k[x, y]$ denote the subspace of polynomials of homogeneous degree 2.

   (a) Show that the action of C_2 given by interchanging x and y defines a representation of C_2 on V_2. Give this in terms of matrices.

   (b) Show that this representation decomposes as a direct sum of some number of copies of $\mathbb{1}$ and ϵ, where ϵ is the non-trivial one-dimensional representation of C_2 (*Hint: look for polynomials fixed or anti-fixed under* C_2)

   (c) Generalise this to all $V_n \subset k[x, y]$, $n \geq 1$.

   *(d) Think about the case of $\mathrm{char}(k) = 2$.

6. Fix $n \geq 1$ and let $V = \mathbb{C}^n$. Let e_i denote the i^{th} basis vector.

   (a) Show there is an n-dimensional representation of S_n given by $e_i \mapsto e_{\sigma(i)}$ under σ.

   (b) Find a non-zero element of V^{S_n}

   (c) Show that V^{S_n} is one-dimensional (*Hint: find a σ for which* $\mathrm{nullity}(\rho(\sigma) - \mathrm{id}) = 1$).

7. (a) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G and $H \leq G$ be a subgroup. Show that there is a representation of H called *restriction*, $\mathrm{Res}^G_H(\rho)$, given by composing $H \hookrightarrow G$ and ρ.

   (b) Let $\rho\colon Q \to \mathrm{GL}(V)$ be a representation of Q, where Q is a quotient of G. Show that there is a representation of G defined by composing the quotient map $G \twoheadrightarrow Q$ and ρ. The resulting representation is called the *inflation* or *lift* of ρ and denoted $\mathrm{Inf}^G_Q(\rho)$.

   (c) Show that $\mathrm{Inf}^G_Q(\rho)$ is irreducible if and only if ρ is.

   (d) Let $f\colon G' \to G$ be any homomorphism. Show that a representation of G defines a representation of G'. This is called *pullback*.

8. The quaternion group Q_8 is defined by $\langle i, j \mid i^2 = j^2 = -1, ij = -ji\rangle$.

   (a) Show that the quaternions can alternatively be thought of a subgroup of $\mathrm{GL}_2(\mathbb{C})$ defined by

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, j \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

   Use this to define a representation ρ of Q_8. Show that it is irreducible.

   (b) Show that the restriction of ρ to $C_4 \cong \langle i \rangle \leq Q_8$ is reducible and splits as a sum of two one-dimensional representations.

9. Show that a one-dimensional representation of a finite abelian group is given by a choice of one-dimensional representation for each factor of its decomposition as a finitely generated abelian groups. In particular, how many one-dimensional $\mathbb{C}$-representations of a finite abelian group A are there?

10. (a) Show that any one-dimensional representation must have abelian image (i.e. the image as a group homomorphism is abelian).

   (b) Show that one-dimensional representations of G are in bijection with one-dimensional representations of $G^{\mathrm{ab}} = G/[G, G]$ (Hint: use 7(b), you may need to look up the properties of the abelianisation).

11. Consider the representation of $\rho\colon C_4 \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\sigma \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (a) Show that ρ has no one-dimensional submodule and so is irreducible.

   (b) Show that the representation $\rho'\colon C_4 \to \mathrm{GL}_2(\mathbb{C})$ defined by the same matrices is reducible. (In this case we say that ρ is irreducible but not *absolutely irreducible*).

   *(c) Show that ρ provides a counter-example to the second part of Schur's lemma holding over an non-algebraically closed field.

2 Sections 3 + 4

1. Given a homomorphism of R-modules $f\colon M \to N$,

   i) $\ker f$ is a submodule of M,

   ii) $\mathrm{im}\, f$ is a submodule of N.

2. Show that $\mathrm{Hom}_k(\rho_1, \rho_2 \oplus \rho_3) = \mathrm{Hom}_k(\rho_1, \rho_2) \oplus \mathrm{Hom}_k(\rho_1, \rho_3)$. Deduce that the same is true for $\mathrm{Hom}_G(-, -)$.

3. Calculate the duals of the n one-dimensional complex representations of C_n.
4. If ρ is an irreducible representation and χ is a one dimensional representation, show that $\chi \otimes \rho$ is also an irreducible representation.
5. Let A be an abelian group and $\rho\colon A \to \mathrm{GL}(V)$ be a representation.
   (a) Show that for any $g \in A$, the linear map $\rho(g)\colon V \to V$ defines an endomorphism of V as $k[A]$-modules.
   (b) Now assume that k is an algebraically closed field. Using Schur's lemma, deduce that any irreducible representation of an abelian group must be one-dimensional.
6. Let M be a module of a ring of the form $R \times S$.
   (a) Show that $(1_R, 0) \cdot M \subseteq M$ is an $R \times S$-submodule of M.
   (b) Show that the map

$$\begin{aligned} M &\longrightarrow (1_R, 0) \cdot M \oplus (0, 1_S) \cdot M \\ m &\longmapsto ((1_R, 0) \cdot m, (0, 1_S) \cdot m) \end{aligned}$$

   is an isomorphism of $R \times S$-modules.
   (c) Deduce that for a simple $R \times S$-module, either R or S must act as zero.
7. Let V be a two-dimensional $k[G]$-module with basis e_1, e_2.
   (a) Show that $V \otimes V$ has a one-dimensional submodule spanned by $(e_1 \otimes e_2 - e_2 \otimes e_1)$ (Hint: suppose $\rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$...).
   (b) Describe the resulting representation $\chi\colon G \to \mathrm{GL}_1(k)$ in terms of matrices.
   *(c) Show that this was all independent of the choice of basis.
8. (a) Show that for any map $f\colon G \to H$, there is an induced ring homomorphism $k[G] \to k[H]$. Show that this is a homomorphism of $k[G]$-modules when g acts on $k[H]$ by multiplication by $f(g)$ (in fact, this is considering $k[H]$ pulled back to G).
   (b) Consider the quotient map $C_4 \to C_2$. The kernel of $k[C_4] \to k[H]$ is a $k[C_4]$-module (why?). Find the matrices describing the corresponding representation of C_4. Show that this decomposes as a sum of two one-dimensional representations.
9. (a) Show that as a ring $k[C_n] \cong k[x]/(x^n - 1)$.
   (b) Show that the ideals of $k[x]$ defined by $(X - a)$,$(X - b)$ are coprime if $a \neq b$.
   (c) Use Chinese remainder theorem to show that as rings

$$\mathbb{C}[C_n] \cong \mathbb{C} \times \ldots \times \mathbb{C}.$$

   (d) Using Lemma 3.15, deduce that there are exactly n isomorphism classes of simple $\mathbb{C}[C_n]$-modules and they are all one dimensional.
   *(e) Find a description of $\mathbb{F}_p[C_p]$. How many maximal ideals does it have? How many isomorphism classes of one-dimensional representations of C_p are there over $\mathbb{F}_p$?

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